Dual Lattices

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Seminar on Lattices and Cryptography

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Outline

1. Duality in algebra
2. Dual lattice
3. Properties of dual lattices
4. Transference Theorems
5. q-ary lattices

Dual of a vector space

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Any linear function $\varphi : V \to \mathbb{R}$ can be represented as a vector $w \in V$ such that

$$\varphi(x) = \langle w, x \rangle \ , x \in V$$

by choosing the vector

$$w = (\varphi(e_1), \ldots, \varphi(e_n)) \ ,$$

where $e_1, \ldots, e_n$ are the canonical basis of $\mathbb{R}^n$.

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- Dual of a module (similarly as for vector spaces previously): Let M be a module over a ring R. The module

$$M^\vee = \{\varphi : M \to R\} = \mathrm{Hom}_R(M, R)$$

is called the **dual module** of M

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- $\mathcal{L}$ is a $\mathbb{Z}$-module
- To every $v \in \mathcal{L}^\vee$ we associate the map (a $\mathbb{Z}$-module homomorphism) $\varphi_v(x) = \langle v, x \rangle$ and by assumption we have $\varphi_v(x) \in \mathbb{Z}$. i.e we can identify both notions via

$$\mathcal{L}^\vee \to \text{Hom}_{\mathbb{Z}}(\mathcal{L}, \mathbb{Z}), v \mapsto (\varphi_v : \mathcal{L} \to \mathbb{Z}, x \mapsto \langle x, v \rangle)$$

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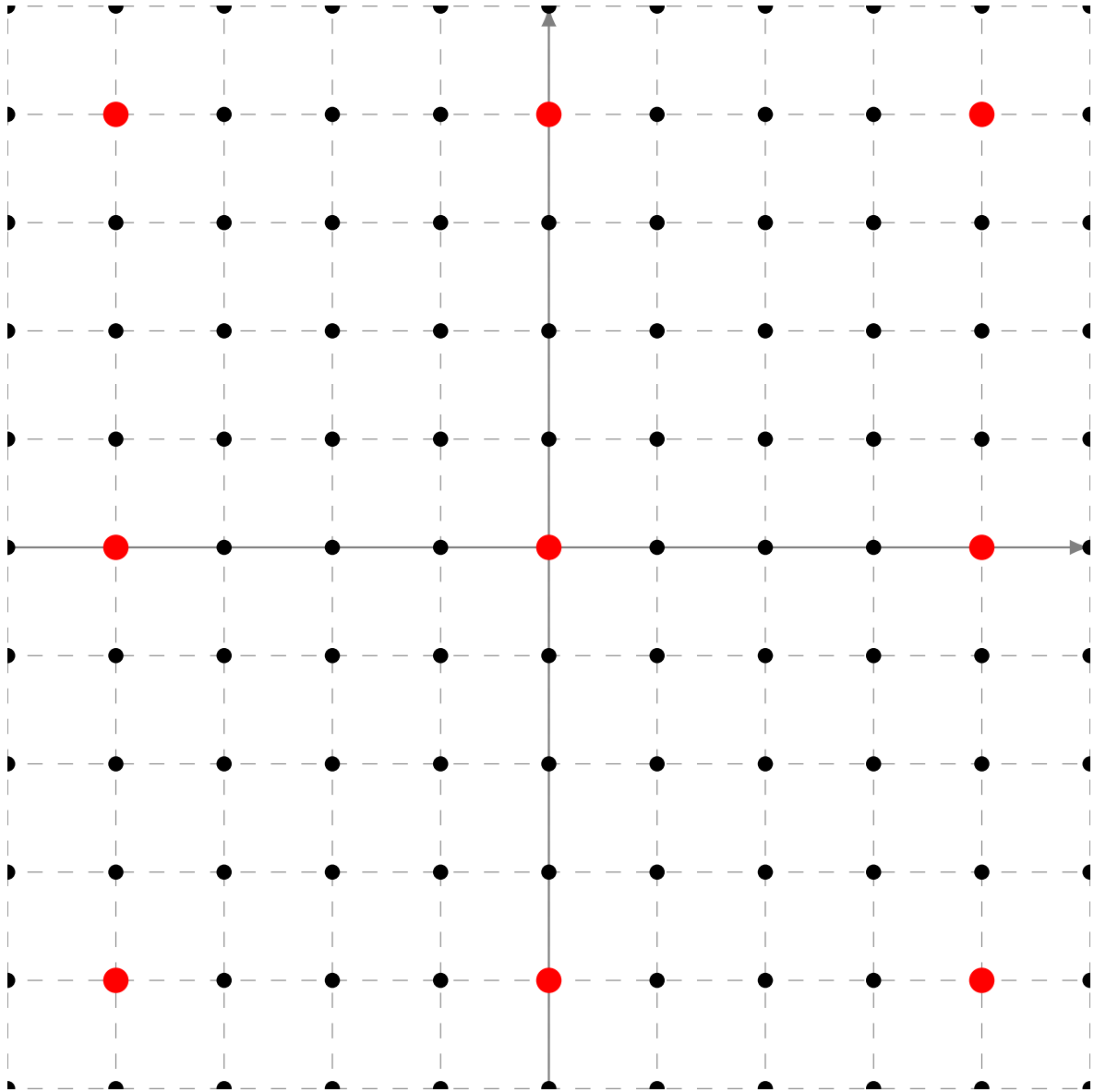
Some questions

- More generally, can we describe a basis of $\mathcal{L}^\vee$ from a basis of $\mathcal{L}$?
- Can we study the geometry of $\mathcal{L}^\vee$ from the geometry of $\mathcal{L}$?

The lattice $2\mathbb{Z}^2$ and its dual $(1/2)\mathbb{Z}^2$



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For every $v \in \mathcal{L}^\vee$ we obtain a partition of the lattice $\mathcal{L}$ into layers by setting

$$\mathcal{L} = \bigcup_{a\in\mathbb{Z}} \{x \in \mathcal{L} : \varphi_v(x) = \langle v, x\rangle = a\}$$

and we set $\{x \in \mathcal{L} : \langle v, x\rangle = a\} =: \mathcal{L}_a$ for $a \in \mathbb{Z}$. Then we have $\mathcal{L} = \bigcup_a \mathcal{L}_a$.

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In conclusion, one has:

$$\begin{aligned} v \in \mathcal{L}^\vee, v \neq 0 \quad &\iff \quad \mathcal{L} \subseteq \{x \in \mathrm{span}_{\mathbb{R}}(\mathcal{L}) : \langle x, v\rangle \in \mathbb{Z}\} \\ &\iff \quad \mathcal{L} \subseteq \mathbb{Z}v' \oplus (\mathbb{R}v)^\perp = \bigcup_{m \in \mathbb{Z}} (mv' + (\mathbb{R}v)^\perp) \end{aligned}$$

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We have the equivalences

$$\begin{aligned} v \in \mathcal{L}^\vee &\iff \langle v, x \rangle \in \mathbb{Z}, \forall x \in \mathcal{L} \\ &\iff B^T v \in \mathbb{Z}^n \\ &\iff v \in (B^{-1})^T \mathbb{Z}^n \end{aligned}$$

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We have proved that $\mathcal{L}^\vee = (B^{-1})^T \mathbb{Z}^n$ and therefore a basis matrix of $\mathcal{L}^\vee$ is

$$B^\vee := (B^{-1})^T \in GL_n(\mathbb{R})$$

An Example

Consider the lattice $\mathcal{L} \subseteq \mathbb{R}^2$ generated by the columns of the matrix

$$B = \begin{bmatrix} 11 & 5 \\ -7 & 1/2 \end{bmatrix}$$

- $v = (v_1, v_2) \in \mathcal{L}^\vee$ iff $(11v_1 + 5v_2 \in \mathbb{Z}) \wedge (-7v_1 + 1/2v_2 \in \mathbb{Z})$
- Therefore we solve the linear system for $x_1, x_2 \in \mathbb{Z}$:

$$(11v_1 + 5v_2 = x_1) \wedge (-7v_1 + 1/2v_2 = x_2)$$

- which is equivalent to

$$(v_1 = 1/81x_1 - 10/81x_2) \wedge (v_2 = 14/81x_1 + 22/81x_2)$$

A basis matrix for $\mathcal{L}^\vee$ is therefore:

$$B = \begin{bmatrix} 1/81 & -10/81 \\ 14/81 & 22/81 \end{bmatrix} = B^{-1}$$

Finding the dual basis, cont'd

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- $\langle D\rangle \subseteq \mathcal{L}^\vee$: Then for any $x = \sum_{i=1}^n \alpha_i b_i \in \mathcal{L}$ with $\alpha_i \in \mathbb{Z}$ we write for all j: $\langle x, d_j\rangle = \sum_{i=1}^n \alpha_i \langle b_i, d_j\rangle = \alpha_i \in \mathbb{Z}$. Hence every $y = \sum_{j=1}^n \beta_j d_j \in \langle D\rangle$ is in $\mathcal{L}^\vee$.

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- $\mathcal{L}^\vee \subseteq \langle D \rangle$: Let $y \in \mathcal{L}^\vee$. In particular $y \in \text{span}_{\mathbb{R}}(B) = \text{span}_{\mathbb{R}}(D)$, so $y = \sum_{i=1}^n \alpha_i d_i$ for $\alpha_i \in \mathbb{R}$. But, $\alpha_j = \sum_i \alpha_i \langle d_i, b_j \rangle = \langle y, b_j \rangle \in \mathbb{Z}$ for all j. This means $y \in \langle D \rangle$.

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The unique basis $D \in \mathbb{R}^{m \times n}$ (allow $m = n$) satisfying

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- In particular for the full rank case:

$$B^\vee = B(B^T B)^{-1} = BB^{-1}(B^T)^{-1} = (B^{-1})^T$$

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For any lattice $\mathcal{L}$ one has $(\mathcal{L}^\vee)^\vee = \mathcal{L}$.

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Proof.

Let B be a basis matrix of $\mathcal{L}$. By the previous theorem a basis of $\mathcal{L}^\vee$ is given by $B^\vee = B(B^T B)^{-1}$. Therefore a basis matrix of $(\mathcal{L}^\vee)^\vee$ is

$$\begin{aligned}(B^\vee)^\vee &= B^\vee((B^\vee)^T B^\vee)^{-1} \\ &= B(B^T B)^{-1}\left((B(B^T B)^{-1})^T B(B^T B)^{-1}\right)^{-1} = B\end{aligned}$$

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Conclusion

Let $\mathcal{L}$ be a lattice (primal) and $\mathcal{L}^\vee$ its dual. Then $\mathcal{L}$ is the dual of $\mathcal{L}^\vee$. Hence the pair $(\mathcal{L}, \mathcal{L}^\vee)$ is called a pair of dual lattices.

How are primal and dual related?

Theorem

Let $\mathcal{L}$ be a lattice. The following hold:

1. $\det(\mathcal{L}^\vee) = \det(\mathcal{L})^{-1}$
2. If $\mathcal{L} \subseteq \mathbb{Z}^m$ then $\mathcal{L}^\vee \subseteq \det(\mathcal{L})^{-2}\mathbb{Z}^m$

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1. $\det(\mathcal{L}^\vee) = \sqrt{\det((B^\vee)^T B^\vee)} = \sqrt{\det(B^T B)^{-1}} = \det(\mathcal{L})^{-1}$
2. Let $\Delta = \det(\mathcal{L})$. Let $B \in \mathbb{Z}^{m\times n}$ basis matrix for $\mathcal{L}$ with dual basis $B^\vee = BG_B^{-1}$ with $G_B = B^T B$. Since B is integral, it is enough to show that $\langle G_B^{-1}\rangle \subseteq \Delta^{-2}\mathbb{Z}^n$. Since G_B is symmetric we have $G_B^T G_B^{-1} = I$. Moreover, $\mathrm{span}_{\mathbb{R}}(G_B) = \mathrm{span}_{\mathbb{R}}(G_B^{-1})$ (because full rank). Therefore, by Theorem of Dual Basis we have $\langle G_B\rangle^\vee = \langle G_B^{-1}\rangle$. Since $G_B \in \mathbb{Z}^{n\times n}$, it follows $\langle G_B\rangle^\vee \subseteq |\det(G_B)|^{-1}\mathbb{Z}^n = \Delta^{-2}\mathbb{Z}^n$.

□

Transference Theorems related to successive minima

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Recall (Successive minima)

For a lattice $\mathcal{L}$ in $\mathbb{R}^m$ of rank $0 < n \leq m$, the ith $(1 \leq i \leq n)$ successive minimum is defined as

$$\lambda_i(\mathcal{L}) = \inf\{\rho > 0 : \dim_{\mathbb{R}}(\mathrm{span}_{\mathbb{R}}(\mathcal{L}) \cap B_m(0, \rho)) \geq i\}$$

i.e. the smallest positive radius of an m-dimensional ball containing i linearly independent vectors of $\mathcal{L}$

Transference Theorems related to successive minima

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Goal question

Consider $\mathcal{L}$ (rank n in $\mathbb{R}^m$) with successive minima $(\lambda_1, \ldots, \lambda_n)$ and $\mathcal{L}^\vee$ with successive minima $(\lambda_1^\vee, \ldots, \lambda_n^\vee)$. Can we 'transfer' knowledge from $(\lambda_1, \ldots, \lambda_n)$ to $(\lambda_1^\vee, \ldots, \lambda_n^\vee)$?

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Remark. In fact a stronger version of Minkowski's Theorem is true: $\lambda_1 \leq \sqrt{\gamma_n} \det(\mathcal{L})^{1/n}$ where γ_n is Hermite's constant in dimension n. Therefore the stronger inequality holds, namely $\lambda_1 \lambda_1^\vee \leq \gamma_n$.

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Example

Consider $\mathcal{L} = 10^6 \cdot \mathbb{Z}^5$. We know that $\mathcal{L}^\vee = 10^{-6}\mathbb{Z}^5$. Then $\lambda_1 = 10^6$ and $\lambda_1^\vee = 10^{-6}$.

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- let $x_1, \ldots, x_n \in \mathcal{L}^{\vee}$ linearly independent such that $\|x_n\| = \lambda_n^{\vee}$ and $\|x_i\| \leq \|x_n\|$ for all $1 \leq i \leq n-1$

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- It follows:

$$1 \leq |\langle v, x_j \rangle| \leq \|v\| \cdot \|x_j\| \leq \lambda_1 \cdot \lambda_n^\vee$$

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More generally, it holds (partial summary)

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Let $\mathcal{L}$ be a lattice of full rank n. Then

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2. Improved bounds can be found in the literature.

q-ary lattices

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Let $q \in \mathbb{Z}_{\geq 3}$ be an odd prime number, $m \in \mathbb{Z}_{\geq 1}$. A lattice $\mathcal{L} \subseteq \mathbb{Z}^m$ is called q-**ary** if

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- $\mathcal{L}$ has full rank m
- q-ary lattices are important for many cryptographic lattice-based schemes/problems, i.e. LWE, SIS, etc.

Typical examples

Let q be an odd prime. Denote $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$ the finite field with q elements. Let $A \in (\mathbb{F}_q)^{m \times n}$ be a matrix, $m \geq n$.

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Definition

$$\Lambda_q(A) = \{y \in \mathbb{Z}^m : y = Ax \pmod{q} \text{ for some } x \in \mathbb{Z}^n\}$$

$$\Lambda_q^{\perp}(A) = \{y \in \mathbb{Z}^m : A^T y \equiv 0 \pmod{q}\}$$

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The lattices $\Lambda_q(A)$ and $\Lambda_q^{\perp}(A)$ are q-ary. Even stronger:

Theorem

The following are equivalent:

1. $\mathcal{L}$ is q-ary
2. $\mathcal{L} = \Lambda_q(A)$ for some A
3. $\mathcal{L} = \Lambda_q^{\perp}(A')$ for some A'

An example for $\Lambda_q(A)$ and $\Lambda_q^{\perp}(A)$

Let $q = 131101$ prime, $m = 3, n = 2$ Let

$$A = \begin{pmatrix} 66612 & 77907 \\ 64603 & 16912 \\ 10733 & 125395 \end{pmatrix}$$

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1. Elements $v \in \Lambda_q(A)$ satisfy:

$$v \equiv \alpha_1 \begin{pmatrix} 66612 \\ 64603 \\ 10733 \end{pmatrix} + \alpha_2 \begin{pmatrix} 77907 \\ 16912 \\ 125395 \end{pmatrix} \pmod{131101}$$

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2. Elements $w = (w_1, w_2, w_3) \in \Lambda_q^{\perp}(A)$ satisfy:
$66612w_1 + 64603w_2 + 10733w_3 \equiv 0 \pmod{131101}$
$77907w_1 + 16912w_2 + 125395w_3 \equiv 0 \pmod{131101}$

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Proposition

1. $\det(\Lambda_q^{\perp}(A)) \leq q^n$ with equality iff $rk_{\mathbb{F}_q}(A) = n$
2. $\det(\Lambda_q(A)) \geq q^{m-n}$ with equality iff $rk_{\mathbb{F}_q}(A) = n$

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Proposition

$$\Lambda_q^\perp(A) = q(\Lambda_q(A))^\vee$$

and

$$\Lambda_q(A) = q((\Lambda_q^\perp(A))^\vee) \ .$$

Moreover, $\det(\Lambda_q^\perp(A)) \cdot \det(\Lambda_q(A)) = q^m$

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1. We first show that $\Lambda_q^{\perp}(A) \subseteq q(\Lambda_q(A))^{\vee}$
   - Let $y \in \Lambda_q^{\perp}(A)$. Thus $A^T y \equiv 0 \pmod{q}$ i.e. $A^T y = qz$ for some $z \in \mathbb{Z}^m$.
   - Let $y' \in \Lambda_q(A)^{\vee}$. Thus $y' = Ax + qz'$ for some vectors $x \in \mathbb{Z}^n, z' \in \mathbb{Z}^m$.
   - Now write

$$\begin{aligned} \langle y, y' \rangle &= \langle y, Ax \rangle + q\langle y, z' \rangle = \langle A^T y, x \rangle + q\langle y, z' \rangle \\ &= q\langle z, x \rangle + q\langle y, z' \rangle \end{aligned}$$

   - This implies $\langle (1/q)y, y' \rangle \in \mathbb{Z}$, so $(1/q)y \in (\Lambda_q(A))^{\vee}$

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   - This implies $\langle (1/q)y, y' \rangle \in \mathbb{Z}$, so $(1/q)y \in (\Lambda_q(A))^{\vee}$
2. Now we show that $\Lambda_q^{\perp}(A) \supseteq q(\Lambda_q(A))^{\vee}$
   - Let $qy \in q(\Lambda_q(A))^{\vee}$. This implies $\langle y, y' \rangle \in \mathbb{Z}$ for all $y' \in \Lambda_q(A)$.
   - In particular this holds for the basis vectors: $\Lambda_q(A)$ being generated by columns of A and vectors $qe_1, \ldots, qe_n$ we have: $A^T y \in \mathbb{Z}^n$ and $qy \in \mathbb{Z}^m$.
   - So: $A^T(qy) = q(A^T y) \equiv 0 \pmod{q}$.

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- Dual bases can be explicitely described from primal bases.
- Duality in Crypto: Many applications using q-ary lattices, applications to LWE, SIS, CVP, etc.

Many thanks for your attention.