NP-hardness of the shortest vector problem

and its relation to the closest vector problem

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Overview

SVP/CVP definition

Definition

The *optimisation* problem SVP is defined on instances $\mathbf{B} \in \mathbb{Z}^{n \times k}$, with the objective to find a non-zero vector $\vec{u} \in \mathcal{L}(\mathbf{B})$ such that

$$\|\vec{u}\| = \lambda(\mathbf{B}).$$

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Definition

The *optimisation* problem CVP is defined on instances $(\mathbf{B}, \vec{v})$, where $\mathbf{B} \in \mathbb{Z}^{n \times k}$ and $\vec{v} \in \mathbb{Z}^n$, with the objective to find a vector $\vec{u} \in \mathcal{L}(\mathbf{B})$ such that

$$\|\vec{u} - \vec{v}\| = \min_{\vec{w} \in \mathcal{L}(\mathbf{B})} \|\vec{w} - \vec{v}\|.$$

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The *optimisation* problem SVP_γ is defined on instances $\mathbf{B} \in \mathbb{Z}^{n \times k}$, with the objective to find a non-zero vector $\vec{u} \in \mathcal{L}(\mathbf{B})$ such that

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Definition

The *promise* problem GapSVP_γ is defined on instances $(\mathbf{B}, d)$, where $\mathbf{B} \in \mathbb{Z}^{n \times k}$ and $d \in \mathbb{R}^+$, with the objective to distinguish between the cases:

$$\texttt{yes} : \lambda(\mathbf{B}) \leq d$$
$$\texttt{no} : \lambda(\mathbf{B}) > \gamma \cdot d$$

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The *promise* problem GapCVP'_γ is defined on instances $(\mathbf{B}, \vec{y}, d)$, where $\mathbf{B} \in \mathbb{Z}^{n \times n}$ is full-rank, $\vec{y} \in \mathbb{Z}^n$, and $d \in \mathbb{R}^+$, with the objective to distinguish between the cases:

$$\texttt{yes} : \exists \vec{z} \in \{0,1\}^n \, \|\mathbf{B}\vec{z} - \vec{y}\| \leq d$$
$$\texttt{no} : \forall \vec{z} \in \mathbb{Z}^k \forall w \in \mathbb{Z} \backslash \{0\} \, \|\mathbf{B}\vec{z} - w\vec{y}\| > \gamma \cdot d$$

Magic Geometry

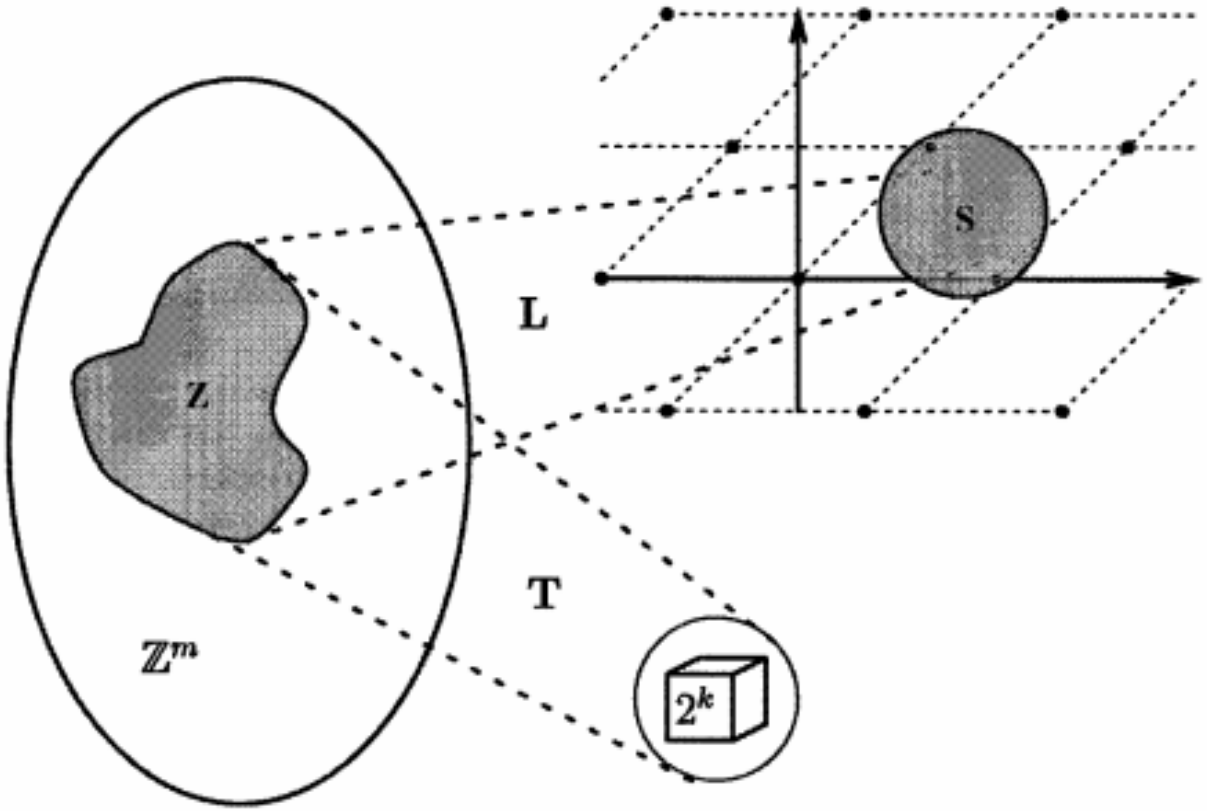
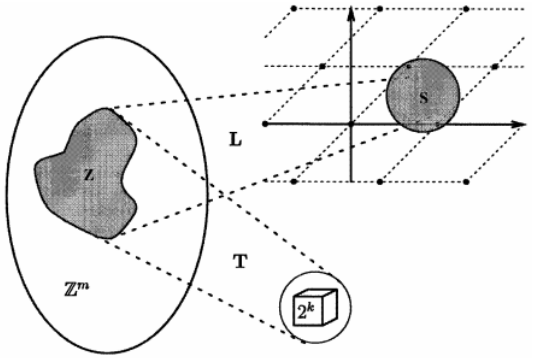


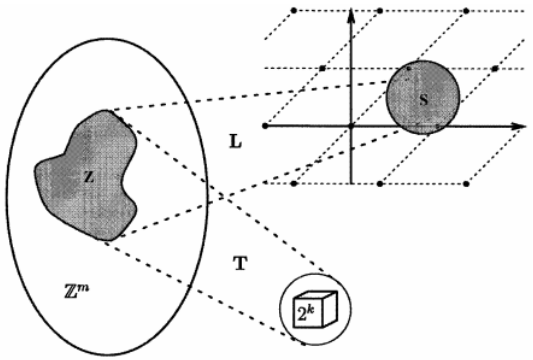
Figure: Source: [Mic01]

Magic Geometry



Parameter: $\tilde{\gamma} \in [1, \sqrt{2})$

Magic Geometry



Parameter: $\tilde{\gamma} \in [1, \sqrt{2})$
Input: $k \in \mathbb{Z}^+$

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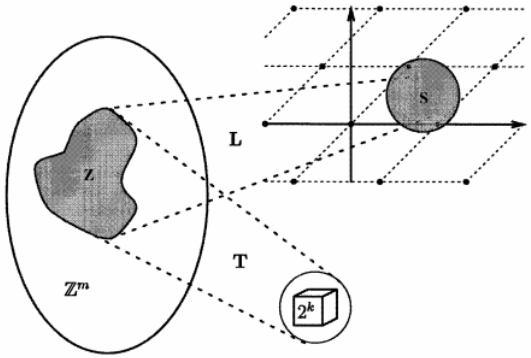

Parameter: $\tilde{\gamma} \in [1, \sqrt{2})$
Input: $k \in \mathbb{Z}^+$
Output: $m, r \in \mathbb{Z}^+, \mathbf{L} \in \mathbb{Z}^{(m+1)\times m}, \vec{s} \in Z^{m+1}, \mathbf{T} \in \mathbb{Z}^{k\times m}$

- $\lambda(\mathbf{L}) > \tilde{\gamma} \cdot r$

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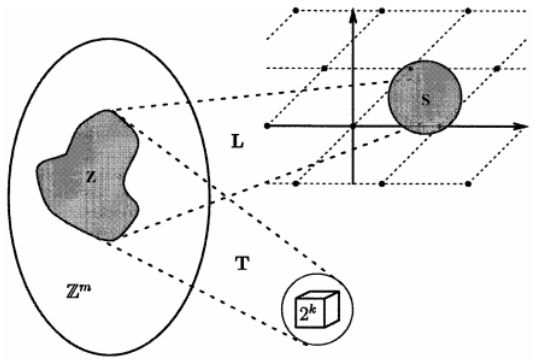

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- $\lambda(\mathbf{L}) > \tilde{\gamma} \cdot r$

- $\forall \vec{x} \in \{0,1\}^k \exists \vec{z} \in \mathbb{Z}^m (\mathbf{T}\vec{z} = \vec{x} \wedge \|\mathbf{L}\vec{z} - \vec{s}\| \leq r)$

Reducing $\mathrm{GapCVP}'_{\gamma'}$ to GapSVP_{γ} [Mic01]

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Magic$_{\tilde{\gamma}}(\mathit{rank}(\mathbf{B}))$: $m, r \in \mathbb{Z}^+, \mathbf{L} \in \mathbb{Z}^{(m+1)\times m}, \vec{s} \in Z^{m+1}, \mathbf{T} \in \mathbb{Z}^{k\times m}$

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$$\mathbf{V} = \left[\begin{array}{c|c} a \cdot \mathbf{BT} & a \cdot \vec{y} \\ b \cdot \mathbf{L} & b \cdot \vec{s} \end{array}\right]$$

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no case: $\left\| \mathbf{V} \begin{bmatrix} \vec{z} \\ w \end{bmatrix} \right\|^2 = (a \cdot \|\mathbf{BT}\vec{z} + w\vec{y}\|)^2 + (b \cdot \|\mathbf{L}\vec{z} + w\vec{s}\|)^2$

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yes case: $\exists \vec{x} \in \{0,1\}^k, \vec{z} \in \mathbb{Z}^m$ such that $\|\mathbf{B}\vec{x} - \vec{y}\| \leq d$ and $\mathbf{T}\vec{z} = \vec{x}$

$$\left\| \mathbf{V} \begin{bmatrix} \vec{z} \\ -1 \end{bmatrix} \right\|^2 = (a \cdot \|\mathbf{B}\vec{x} - \vec{y}\|)^2 + (b \cdot \|\mathbf{L}\vec{z} - \vec{s}\|)^2$$

SetCover definition

Definition

The *promise* problem $\mathrm{SetCover}_c$ is defined on instances $(\mathcal{U}, S_1, \ldots, S_m, K)$, where $\mathcal{U}$ is a set, $S_1, \ldots, S_n \subseteq \mathcal{U}$, and $K \in \mathbb{Z}^+$, with the objective to distinguish between the cases:

`yes` :There is a collection of K pairwise disjoint S_i's whose union is $\mathcal{U}$

`no` :There is no collection of less cK S_i's whose union is $\mathcal{U}$

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Theorem ([Bel+93])

For any constant $c \geq 1$, $\mathrm{SetCover}_c$ is NP-hard.

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yes case: $\exists I \subseteq [m] \bigcup_{i \in I} S_i = \mathcal{U}$ and thus $\vec{a} = \chi_I \in \{0,1\}^m$ shows $O_1 \leq \sqrt{K}$

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- $\bigcup \{S_i \mid a_i \neq 0\} = \mathcal{U}$, then $O_w \geq \|\vec{a}\| \geq \sqrt{cK} = \gamma'\sqrt{K}$
- One of the first $|\mathcal{U}|$ coordinates is not 0 and thus $O_w \geq \sqrt{L} = \gamma'\sqrt{K}$

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- One of the first $|\mathcal{U}|$ coordinates is not 0 and thus $O_w \geq \sqrt{L} = \gamma'\sqrt{K}$

Thus $(\mathbf{B}, \vec{b_0}, \sqrt{K})$ is an equivalent $\mathrm{GapCVP}'_{\gamma'}$ instance

Reducing GapSVP_γ to GapCVP_γ

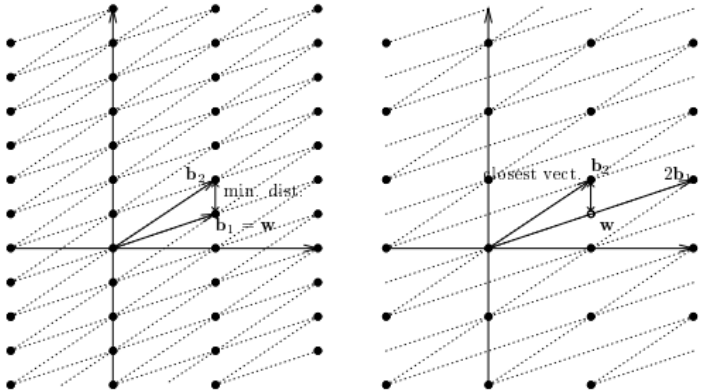
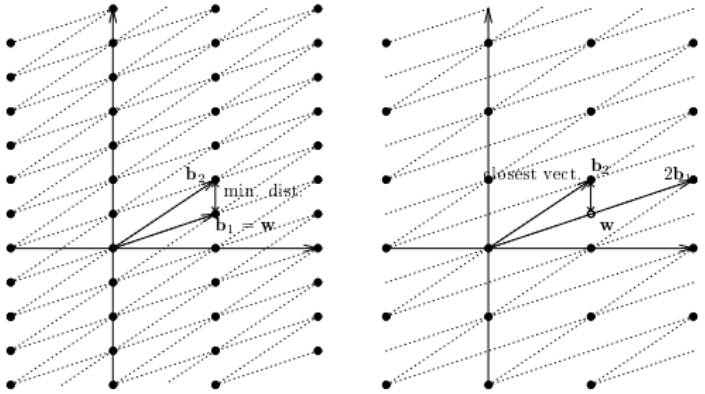


Figure: Source: [GoI+99]

Input: GapSVP_γ instance $(\mathbf{B}, d)$, where $\mathbf{B} = [\vec{b}_1, \ldots, \vec{b}_n]$

Reducing GapSVP_γ to GapCVP_γ



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Input: GapSVP_γ instance $(\mathbf{B}, d)$, where $\mathbf{B} = [\vec{b}_1, \ldots, \vec{b}_n]$

Define for all j: $\mathbf{B}^{(j)} = [\vec{b}_1, \ldots, \vec{b}_{j-1}, 2\vec{b}_j, \vec{b}_{j+1}, \ldots, \vec{b}_n]$

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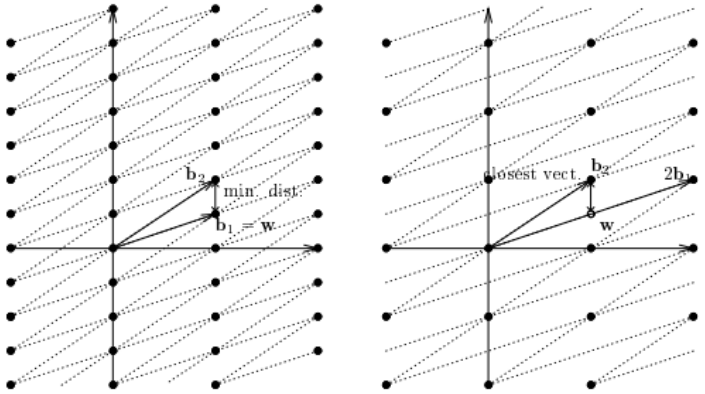

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yes case: Let $\mathbf{B}\vec{x} = \vec{v}$ be such that $\|\vec{v}\| \leq d$, then at least one x_j is odd and $\vec{v} + \vec{b}_j \in \mathcal{L}(\mathbf{B}^{(j)})$

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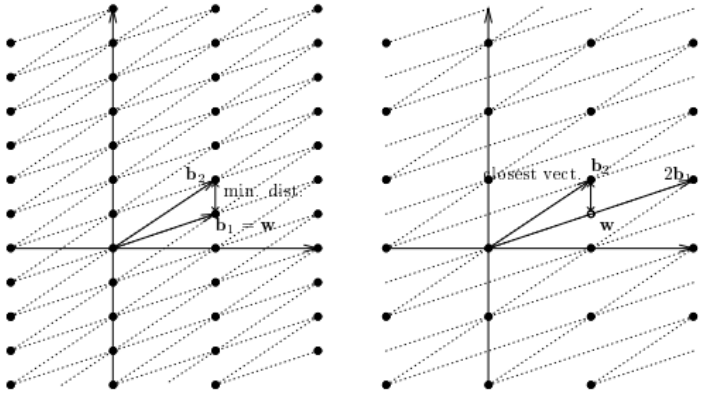

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Input: GapSVP_γ instance $(\mathbf{B}, d)$, where $\mathbf{B} = [\vec{b}_1, \ldots, \vec{b}_n]$

Define for all j: $\mathbf{B}^{(j)} = [\vec{b}_1, \ldots, \vec{b}_{j-1}, 2\vec{b}_j, \vec{b}_{j+1}, \ldots, \vec{b}_n]$

yes case: Let $\mathbf{B}\vec{x} = \vec{v}$ be such that $\|\vec{v}\| \leq d$, then at least one x_j is odd and $\vec{v} + \vec{b}_j \in \mathcal{L}(\mathbf{B}^{(j)})$

no case: For any $\vec{b}_j$ and any $\vec{u} \in \mathcal{L}(\mathbf{B}^{(j)})$, $\vec{v} = \vec{u} - \vec{b}_j \in \mathcal{L}(\mathbf{B})$ thus $\left\|\vec{u} - \vec{b}_j\right\| \geq \gamma d$

Reducing GapSVP_γ to GapCVP_γ

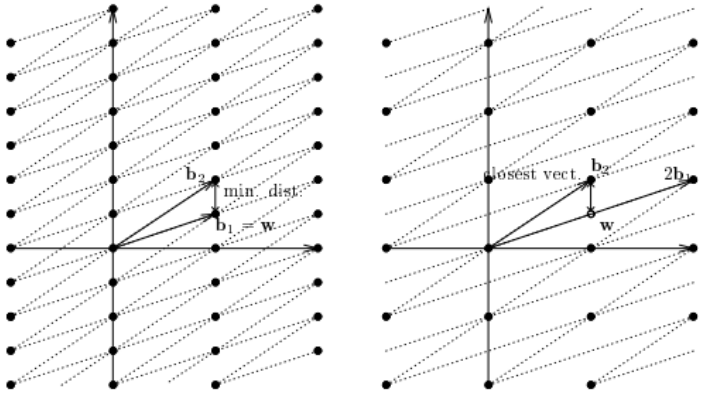

Figure: Source: [Gol+99]

Input: GapSVP_γ instance $(\mathbf{B}, d)$, where $\mathbf{B} = [\vec{b}_1, \ldots, \vec{b}_n]$

Define for all j: $\mathbf{B}^{(j)} = [\vec{b}_1, \ldots, \vec{b}_{j-1}, 2\vec{b}_j, \vec{b}_{j+1}, \ldots, \vec{b}_n]$

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no case: For any $\vec{b}_j$ and any $\vec{u} \in \mathcal{L}(\mathbf{B}^{(j)})$, $\vec{v} = \vec{u} - \vec{b}_j \in \mathcal{L}(\mathbf{B})$ thus $\left\|\vec{u} - \vec{b}_j\right\| \geq \gamma d$

Thus the OR of $\mathrm{GapCVP}_\gamma(\mathbf{B}^{(j)}, \vec{b}_j, d)$ is equivalent to $\mathrm{GapSVP}_\gamma(\mathbf{B}, d)$

Thanks!

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